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Review Article

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Approximation of the Mittag-Leffler Functions by Elementary Functions with Physics Applications

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Abstract
In this paper we give approximations to the Mittag-Leffler functions in terms of elementary functions using different methods. This allowed us to establish a practical method we called integerization principle. This principle states that many fractional nonlinear oscillators may be solved by means of the solution to some integer-order Duffing oscillator equation. The accuracy of the obtained results is illustrated in concrete examples. Formulas for estimating the errors in the approximations are also provided.

Keywords: Fractional Oscillator, Caputo Derivative, Nonlinear Fractional Oscillator, Duffing Equation, Fractional Pendulum, Fractional Van Der Pol Equation, Duffing, Mathieu Fractional Oscillator

Introduction

Fractional calculus generalizes the differential and integral operators to non-integer orders, for this reason it is also called arbitrary order calculation. This idea is as old as calculus itself, but its development has been largely conditioned by the absence of a physical interpretation and convincing geometry and also by the numerous definition proposals. We could say that it did not have a real development until the second half of the 20th century, that is why we find here a classical and at the same time modern branch of mathematics [1-5].

Currently, a large number of articles are published on the subject, and we find applications in most of the sciences, this is because fractional operators are nonlocal operators, that is, what occurs at a point depends on an average over an interval containing the point, and this makes fractional calculus an exceptional tool for non-local phenomena such as ecological processes such as accumulation of metals, problems of population evolution, problems of radiation, economy, etc. They also play a very important role in relaxation processes such as those associated with viscoelastic materials [6-10].

In 1969, the Italian mathematician and physicist Michele Caputo gave a new definition derived from fractional order that allowed to interpret physically the initial conditions of the increasingly numerous applied problems. Caputo (1969) defined the fractional derivative as

$$D_t^\alpha x(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{x''(S)ds}{(t-s)^{\alpha-1}}, 1<\alpha<2 \quad (1)$$

$$D_t^\alpha x(t) = x''(t) \ when \ \alpha = 2 \quad (2)$$

$$D_t^\alpha x(t) = x'(t) \ when \ \alpha = 1 \quad (3)$$

In 1974, the first international conference on Fractional Calculus was held in Connecticut, which served as a stimulus to numerous publications. The second conference took place in 1984 in Scotland, and the third in 1989 in Tokyo. Nowadays it is difficult to find a field of science or engineering that does not consider concepts of the Fractional Calculus, and every year there are several events that show it.

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Most nonlinear differential equations do not admit an analytical procedure that describe its solution. For this reason it is necessary to resort to numerical methods such as Runge Kutta, Newton, Euler etc., generally developed in software tools for applications in mathematics and engineering; these methods are a quite practical solution tool that allows approaching in an approximate way the solution of differential equations nonlinear; another resource contemplated when describing the solution of this type of equations refers to the qualitative information of the general behavior of the solutions that are actually obtains without solving these equations, basically through methods and geometric analysis. The interest of studying nonlinear differential equations lies mainly in that most physical systems, whether electrical, magnetic, biological, chemical, geological, economic, etc., present a non-linear behavior by nature; the procedure of linearizing the equations segments the knowledge about the behavior of systems around an equilibrium point, while the study of systems by means of nonlinear theory makes it possible to be aware of the behavior of the system at all points within which it is defined; another reason to study nonlinear systems and differential equations that describe them lies in the existence of natural phenomena and surprising mathematical representations that have no place in linear theory

Suppose we are given that

$$D_t^{\alpha}x(t)+\omega_0^2x(t)=0,\ x(0)=x_0\ and\ \dot{x}'(0)=x_0 \tag{4}$$

The exact solutuion to the i.vp. (2) reads

$$x(t)=x_0E_{\alpha}(-\omega_0^2t^{\alpha})+\dot{x}_0tE_{\alpha,2}(-\omega_0^2t^{\alpha}) \tag{5}$$

where $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha,2}(-t^{\alpha})$ are the Mittag-Leffler functions. These functions are de ned as follows:

$$E_{\alpha}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha\omega+1)} \tag{6}$$

$$E_{\alpha,\beta}(z)=\sum_{\omega=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)} \tag{7}$$

Our aim is to approximate the Mittag-Le er functions $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha,2}(-t^{\alpha})$ by means of elementary functions. We also will apply the obtained results in the solution of fractional differential equations. The idea is to replace the fractional ode by means of some suitable integer-order oscillator. We will call this method the Integerization Principle.

Approximation of Mittag-Leffler Functions by Means of Elementary Functions

Assume that 1 < 2. Let us consider the Mittag-Le er function $E_{\alpha}(-t^{\alpha})$. To begin with, let α = 1.9. The function $E_{1.9}(-t^{1.9})$ is plotted in Figure 1.

From that figure we see that the function $E_{1.9}(-t^{1.9})$ behaves like a damped oscillator. So, we expect that this function may be approximated by means of the solution to some damped linear oscillator

$$\ddot{x}+2\varepsilon\dot{x}+\left(\omega^2+\varepsilon^2\right)x=0,\ x(0)=1\ and\ x'(0)=0 \tag{8}$$

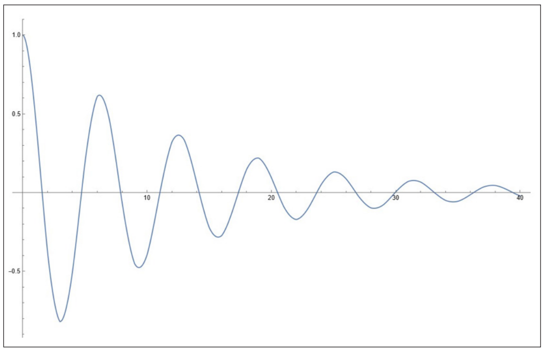

Figure 1: The Function $E_{1.9}(-t^{1.9})$

Let us examine several possibilities for the approximation. Choose some positive number T, say $T = 100$:

First Method.

We look for suitable positive numbers " and w such that $E_{\alpha}(-t^{\alpha}) \approx x_{\varepsilon,\omega}(t)$: = exp(-εt) cos($wt$) for $0 \le t \le 100$

We choose the numbers ε and w so that

$$F(\varepsilon,w)=\min_{\varepsilon,w}\max_t|E_{\alpha}(-t^{\alpha})-x_{\varepsilon,w}(t)|$$

Making use of interpolation techniques based on Chebyshev approximation theories, we obtained the following experimental formula:

$$E_{\alpha}(-t^{\alpha})\approx\exp\left(-(2-\alpha)\left(\frac{14}{25}+\frac{24}{23\left(1+\frac{21\alpha}{10}\right)}\right)^{\alpha-1}\right)\cos\left((\alpha-1)\left(\frac{117}{50(1+e\alpha)}+\frac{53}{22}\right)^{2-\alpha}t\right) \tag{9}$$

so that

$$\varepsilon=(2-\alpha)\left(\frac{14}{25}+\frac{24}{23\left(1+\frac{21\alpha}{10}\right)}\right)^{\alpha-1} \tag{10}$$

and

$$w=(\alpha-1)\left(\frac{117}{50(1+e\alpha)}+\frac{53}{22}\right)^{2-\alpha} \tag{11}$$

Suitable values are presented in Table 1.

Table 1: The approximation $E_{\alpha}(-t^{\alpha}) \approx x_{\varepsilon,\omega}(t)$: = exp(-εt) cos($wt$) for $0 \le t \le 100$

α	ε	w	Error
1	1	0	0
1:05	0:944246	0:142712	0:04
1:1	0:888087	0:268428	0:062
1:15	0:831789	0:379059	0:08
1:2	0:775585	0:476261	0:093
1:3	0:664268	0:635963	0:105
1:4	0:55552	0:757079	0:11
1:5	0:4504	0:847047	0:01
1:55	0:399457	0:88224	0:094
1:6	0:3497	0:911754	0:09
1:65	0:301193	0:936147	0:08

1:7	0:253991	0:955919	0:08
1:75	0:208135	0:971521	0:07
1:8	0:163662	0:983361	0:063
1:85	0:120596	0:991809	0:052
1:9	0:0789572	0:9972	0:039
1:95	0:0387563	0:999838	0:039
2	0	1	0:022

Conclusions

We gave approximate expression for the Mittag-Leffler functions in terms of elementary functions making use of several approaches. This allowed us to establish a method we called integerization principle. We demonstrated in concrete examples the effectiveness of the proposed method. We also may solve other fractional nonlinear oscillator using the integerization principle [16-19].

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